

Small Volumes in Compactified String Theory

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We discuss some of the classical and quantum geometry associated to the degeneration of cycles within a Calabi-Yau compactification. In particular, we focus on the definition and properties of quantum volume, especially as it applies to identifying the physics associated to loci in moduli space where nonperturbative effects become manifest. We discuss some unusual features of quantum volume relative to its classical counterpart.

12/96

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1. Introduction

Our understanding of quantum geometry — the geometrical structure underlying quantum string theory — has deepened significantly over the past few years. From our realization of new geometrical properties associated with an extended object even at the classical level [1] [2], to the numerous new properties [3] which the recent progress in nonperturbative string theory has allowed us to discern, quantum geometry has shown itself to be a remarkably rich structure. Even with the impressive progress that has been made, though, there are still some rather basic aspects which have not been fully understood. In this letter we focus on one such property which surrounds an aspect of *quantum volume*. This is a topic discussed at some length in [4] with an interesting follow up being [5].

A central theme running through many of the most important recent developments is the study of string compactifications in the vicinity of degeneration points in moduli space. At such points, degrees of freedom which normally have inconsequential effects on low energy dynamics play a dominant physical role. It is clearly important, then, to understand where such points actually occur in the quantum mechanically corrected moduli space. The relationship with quantum volume arises because these points are associated with the collapse of nontrivial cycles to “zero quantum volume”; this oft-used phrase certainly deserves meaningful definition and study.

Whenever discussing the extension of a classical concept to the quantum domain there is an inherent ambiguity as many quantum concepts can have the same classical limit. Without specifying a physical incarnation, the subtleties involved in defining quantum volume might amount to nothing more than semantics. In [4], fundamental string instantons which wrap around holomorphic spheres were used as a probe of two-cycle volumes within a Calabi-Yau manifold. This resulted in some interesting observations regarding the identification of special points in and the overall structure of the quantum Calabi-Yau moduli space; these results played a role in [6] and [7], for instance.

Two questions were raised but left unanswered in [4]. The first question, as we discuss below, concerns the properties of the quantum volume of spheres involved in flop/conifold transitions away from boundaries in moduli space. The second question concerns the extension of the analysis in [4] to address the quantum volume of cycles whose dimension is greater than two. In [4] no procedure was proposed beyond the seemingly reasonable assumption that if lower dimensional submanifolds within some chosen cycle have nonzero volume, then the chosen cycle itself must have nonzero volume as well. Without a direct

physical probe of higher dimensional cycles, it was difficult to proceed further. In this paper, using more recent developments, we are able to address each of these issues and reach conclusions that are rather different than what might have been anticipated.

With regards the first question, we explicitly calculate the quantum volume of rational curves in flop/conifold transitions. In [4] this was done while holding all other moduli at infinity, and the result obtained was zero. It was speculated [4] that zero volume — something a bit orthogonal to the increasingly vague notion of there being a minimal distance in string theory — was an artifact of pinning the other moduli at infinity in this manner and that such rational curves would pick up nonzero volume at interior points on the transition locus. Circumstantial evidence in favor of this speculation was given in [5] by studying the simpler case of blown-down orbifolds. We find here, though, that the flop case is different from the orbifold and maintains zero quantum volume all along the transition locus in the moduli space. We note that this is also relevant for the Kähler side of conifold transitions as these occur at the center of flop transitions.

With regards the second question, the discovery of higher dimensional structures in the form of D -brane degrees of freedom does provide us now with such a physical probe of higher dimensional cycles. Part of our purpose is to study some aspects of the picture which emerges if we take wrapped D -brane masses as the operational definition of quantum volume. It is worth noting that as is already apparent from string dualities such as $R \rightarrow 1/R$, the geometry which emerges in studying some configuration depends at least in part on *which* probe one uses. Thus, there is probably no unique notion of quantum geometry but rather a spectrum of possibilities associated with the different ways it is accessed. We note, for instance, that using wrapped D -brane configurations is different from using D -brane scattering as probes of sub-stringy geometry [8], and understanding the detailed relation between the two would be valuable.

Using this approach we find a general picture in which contrary to one's classical intuition the assumption made in [4] is not true: in quantum geometry the collapse of a cycle B to a point, with zero quantum volume, does *not* necessarily imply that subcycles of B with lower dimension are necessarily squeezed to zero quantum volume as well. This clearly has bearing on the identification of resulting zero mass states at the degeneration point since branes wrapping the lower dimensional cycles will not become massless. Similarly, strings arising from wrapping a p -brane around such $p-1$ -cycles will not become tensionless, as well. Such investigations also allow us to clarify the mirror Kähler interpretation of complex structure degenerations of a Calabi-Yau. The rough statement that the mirror of

collapsing S^3 's is collapsing S^2 's is generally incomplete since all of the even dimensional cycles on the mirror side can be involved [9] [10] [11]. A by-product of our discussion is a procedure for making such identifications precise.

In section 2 we present an explicit calculation of the volume of rational curves involved in flop or conifold transitions in a Calabi-Yau three-fold. This mirror symmetry calculation measures volumes as probed by fundamental string instantons. In section 3 we discuss various ways of defining volume more generally and describe the approach of using wrapped D -brane configurations. For two-branes, this makes direct contact with the calculation of section 2. In section 4 we outline the general picture which emerges from this approach. We first apply our general picture to the case of flop/conifold transitions, affirming our explicit calculations of section 2. We then discuss how on general grounds, the collapse of a higher dimensional cycle does not entail the collapse of lower dimensional subcycles, illustrating a novel feature of quantum geometry.

2. Quantum Volumes of Flops and Conifolds

Classical geometry tells us that topology changing flop and conifold transitions involve S^2 's which shrink to zero volume and are then replaced either with other S^2 's or with S^3 's. Quantum mechanically one can imagine that this classical statement is modified in some manner. For instance, transitions might occur at non-zero classical volume which turns out to have zero quantum volume; or non-zero quantum volumes might play a role. For the case of flop transitions, this was first studied in [4] along a calculationaly amenable locus in Calabi-Yau moduli space: a one-parameter subspace along which only the volume of the shrinking S^2 changes while all other Calabi-Yau Kähler moduli are held fixed at infinity. By using mirror symmetry, the Picard-Fuchs equation along the mirror of this locus, which governs the behavior of periods of the holomorphic three-form Ω was found to be

$$\left\{ \left(z \frac{d}{dz} \right)^2 - z \left(z \frac{d}{dz} \right)^2 \right\} \left(\int_{\gamma} \Omega \right) = 0. \quad (2.1)$$

The regular and logarithmic monodromy solutions yield the complex volume for the S^2

$$B + iJ = \frac{1}{2\pi i} \log(z). \quad (2.2)$$

In this expression, z is a local moduli space coordinate normalized so that $z = 1$ is the flop point. We see, then, that at $z = 1$ the quantum volume, like the classical volume, vanishes identically.

Although attaining zero quantum volume, the expectation of [4] was that this was an artifact of holding all other Kähler moduli at infinity. As mentioned above, at least in the case of orbifold singularities, there are examples which bear this out [5]. Presently, we explicitly study the flop/conifold case. One expects this case to be significantly more difficult than those studied in [4] and [5] since the locus of interest is generally along an interior transition wall in the moduli space, rather than along a boundary divisor.

To set up the calculation, let M be a Calabi-Yau threefold with mirror W . Following the by now standard discussion of [12] [13], we can explicitly evaluate the quantum corrected value of the complexified Kähler class of M by doing *classical* geometrical calculations on W and invoking the mirror map. Abstractly, the nonlinear sigma model on M — for fixed complex structure — depends on the complexified Kähler class K . Although the latter is not a direct physical observable of a given theory, knowledge of a sufficiently robust set of correlation functions is enough data to determine K on M . The volume of a two-cycle C on M (which we will always take to be an S^2) is then given by $\int_C K$ where K is the complexified Kähler form. The latter is given by

$$K = t_i e^i \tag{2.3}$$

with

$$t_i = \frac{\int_{\gamma_i} \Omega}{\int_{\gamma_0} \Omega}, \tag{2.4}$$

Ω being the holomorphic three-form on W , the γ_i , $i = 1, \dots, h_W^{2,1}$ being a basis of three-cycles with log-monodromy periods at large complex structure, and γ_0 being a three-cycle with regular period. As mirror symmetry aligns $H_2(M, \mathbb{Z})$ with the log-period monodromy subspace of $H_3(W, \mathbb{Z})$ (with a similar statement for their respective cohomological duals), by suitable change of basis, mirror symmetry identifies the integral generators e_i of $H_2(M, \mathbb{Z})$ with the γ_i . Thus, since C can be written as $C = a^j e_j$ and with e_j being a dual cycle to the class e^j , we have $\int_C K = t_i a^i$. Without loss of generality, then, we take our curves C to be amongst the e_j which then have volume given by t_j as given in (2.4).

To compute quantum volumes of two-dimensional cycles on M , therefore, we need to know the log-monodromy periods on its mirror W . In this section we will explicitly evaluate these periods for two two-parameter examples which have Calabi-Yau phases related by a flop transition. In the first example we exploit an observation made in [14] that flop/conifold transitions of a particular special sort are subject to strong renormalization resulting in a significant shift in the location of the flopping wall. In particular, the wall is

pushed out to a toric boundary divisor, thus fortuitously bringing the explicit calculation under full analytic control. In the second example, we consider a more standard type of flop transition involving an interior transition wall. We analyze two-cycle volumes using a perturbation scheme. The discussion is naturally phrased in terms of toric geometry (see, e.g. [15] and [2] for an introduction to this subject).

2.1. Small Resolution of a Singular Quintic in a \mathbb{P}^4

Consider a family of quintics in a \mathbb{P}^4 . This family has one Kähler parameter, i.e. the overall size of the manifold and 101 complex structure parameters. If we deform the complex structure parameters so that the quintic develops conical singularities as in [16], [17] and then perform a small resolution by blowing up along the \mathbb{P}^2 containing the singular points, we will get a new family of Calabi-Yaus with 2 Kähler parameters and 86 complex structure parameters. The first Kähler parameter is the original overall size of the manifold and the second controls the size of the sixteen homologous S^2 's introduced as the result of the blow-up. This is the two-parameter model we want to study. As discussed in [14], this two-parameter model has two phases related by flops — with some rather unusual features which will aid our calculation below.

The mirror of this family has the following toric description. The fan for the original \mathbb{P}^4 is spanned by $u_1, \dots, u_5 \in \mathbb{C}^4$, where

$$u_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad u_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad u_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad u_4 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \quad u_5 = \begin{pmatrix} -1 \\ -1 \\ -1 \\ -1 \end{pmatrix} \quad (2.5)$$

Blowing up adds $u_6 = u_4 + u_5$ to the polytope. Thus a typical manifold W in this two-parameter family is a toric variety V_Δ , where Δ is the polytope with vertices u_1, \dots, u_6 .

As we will see shortly, to understand the moduli space of complex structures of W one needs to consider the total space of the canonical line bundle on W . As a toric variety, this space is built using polytope Δ^+ in \mathbb{C}^5 with vertices given by the origin O of \mathbb{C}^5 and

$$v_0 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad v_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad v_3 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad v_4 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \quad v_5 = \begin{pmatrix} 1 \\ -1 \\ -1 \\ -1 \\ -1 \end{pmatrix}, \quad v_6 = \begin{pmatrix} 1 \\ -1 \\ -1 \\ -1 \\ 0 \end{pmatrix} \quad (2.6)$$

To analyze the phase structure of our moduli space we need the kernel of χ , which is a 5 by 7 matrix with columns v_0, \dots, v_6 . One can check that a possible basis for the kernel is $\begin{pmatrix} \xi \\ \eta \end{pmatrix}$, where

$$\begin{pmatrix} \xi \\ \eta \end{pmatrix} = \begin{pmatrix} -5 & 1 & 1 & 1 & 1 & 1 & 0 \\ -1 & 0 & 0 & 0 & 1 & 1 & -1 \end{pmatrix} \quad (2.7)$$

The secondary fan of W is generated by the two-dimensional column-vectors of (2.7), and is shown in Fig. 1.

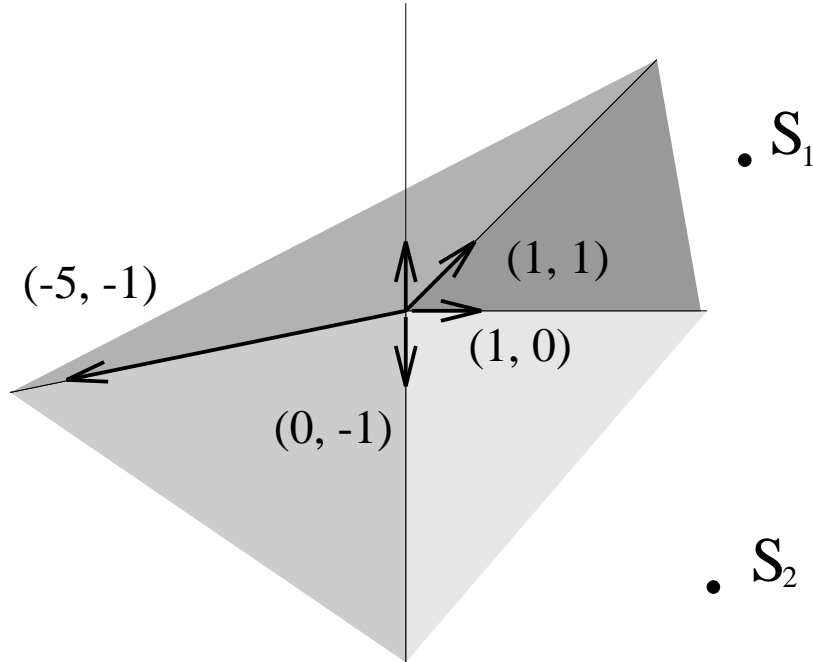


Figure 1. The secondary fan of a blown-up quintic in \mathbb{P}^4

The deep interior of phase I corresponds to the neighborhood of the infinite size point of M (symbolically represented by the point S_1 in Fig. 1) or the neighborhood of infinite complex structure of its mirror W . Likewise, the deep interior of phase II (the neighborhood of S_2 in Fig. 1) has a similar significance for a different Calabi-Yau \widetilde{M} , birational to the first. As explained in [14], \widetilde{M} can be obtained from M by flopping a complex curve. The mirror of this transition involves deforming the complex structure of W until a three-cycle vanishes. At this point in the moduli space of complex structures the space becomes singular. One can then continue to deform the complex structure thereby passing once again to a smooth complex structures on W . So, one possible approach to

the problem of computing the quantum volume of a two-cycle $\mathcal{C} \in H_2(M, \mathbb{Z})$ would be to explicitly identify its mirror three-cycle $\gamma \in H_3(W, \mathbb{Z})$ and evaluate $\int_\gamma \Omega$ with properly normalized three-form Ω . This is technically difficult. Instead, we can use the following familiar but less direct method.

- All periods $\int_\gamma \Omega$ satisfy Picard-Fuchs equations. Gel'fand and collaborators [18] showed how to write such partial differential equations for any toric variety. For our two-parameter example the Picard-Fuchs system will consist of the following two equations:

$$\{\theta_1^3(\theta_1 - \theta_2) - x(4\theta_1 + \theta_2 + 1) \cdots (4\theta_1 + \theta_2 + 4)\}\Phi(x, y) = 0 \quad (2.8)$$

and

$$\{\theta_2^2 + y(4\theta_1 + \theta_2 + 1)(\theta_1 - \theta_2)\}\Phi(x, y) = 0, \quad (2.9)$$

where

$$\theta_1 = x\partial_x, \quad \theta_2 = y\partial_y \quad (2.10)$$

and x and y are variables on the space of complex structures of W such that $x = y = 0$ is the infinite complex structure point.

- Find a solution which is single-valued around $x = y = 0$ and two solutions with log-monodromy at $x = y = 0$. From the previous section we know that the latter will be of the form $\int_{\gamma_1} \Omega$ and $\int_{\gamma_2} \Omega$, where γ_1 and γ_2 are mirrors of the two-cycles. Methods to do this were developed in [19] [5]. The single-valued solution is

$$\Phi_0(x, y) = \sum_{m \geq 0} \sum_{n \leq m} \frac{(4m + n)!}{(m!)^3 (n!)^2 (m - n)!} x^m (-y)^n, \quad (2.11)$$

while the log-monodromy solutions are

$$\Phi_1(x, y) = -\log(x)\Phi_0 + \dots \quad (2.12)$$

$$\begin{aligned} \Phi_2(x, y) &= -\log(-y)\Phi_0 \\ &+ \sum_{m \geq n \geq 0} \frac{(4m + n)!}{(m!)^3 (n!)^2 (m - n)!} x^m (-y)^n [2\Psi(n + 1) - \Psi(4m + n + 1) - \Psi(m + 1 - n)] \\ &- \sum_{m \geq 0} \sum_{n \geq m+1} \frac{(n - m - 1)!(4m + n)!}{(m!)^3 (n!)^2} (-x)^m y^n. \end{aligned} \quad (2.13)$$

These power series converge in some neighborhood of $x = y = 0$. Note that when we hold the overall size of the manifold at infinity, i.e. when $x = 0$, $\Phi_0(0, y) \equiv 1$, $\Phi_1(0, y) \equiv \infty$ as it should and Φ_2 specializes to

$$\Phi_2(0, y) = \log \frac{y-1}{y}, \quad (2.14)$$

which is precisely the expression found in [14] for the size of the flopped curve when the other modulus is held at infinity. As explained there, the effects of string instantons push the conifold point to $y = \infty$ and thus at the conifold Φ_2 (and therefore the quantum area of the complex curve) is equal to zero. Since we now have a full solution, we are in position to check the conjecture that this zero volume is an artifact of the other Kähler modulus being infinite.

The component of the discriminant locus which passes through the point $x = 0, y = \infty$ is simply given locally by $|x| < \varepsilon, y = \infty$ and to study the behavior of the quantum volume, we simply need to analytically continue the above expressions for Φ_0 and Φ_2 to the neighborhood of $x = 0, 1/y = 0$. To do this, let's write Φ_0 and Φ_2 as Barnes integrals

$$\Phi_0 = \sum_{m \geq 0} \frac{x^m}{(m!)^3} \frac{1}{2\pi i} \int_C \frac{\Gamma(z-m)\Gamma(4m+z+1)}{\Gamma(z+1)^2} y^z dz \quad (2.15)$$

and

$$\Phi_2 = \sum_{m \geq 0} \frac{(-x)^m}{(m!)^3} \frac{1}{2\pi i} \int_C \frac{\Gamma(z-m)\Gamma(-z)\Gamma(4m+z+1)}{\Gamma(z+1)} (-y)^z dz, \quad (2.16)$$

where C is the contour in the z -plane shown in Fig. 2.

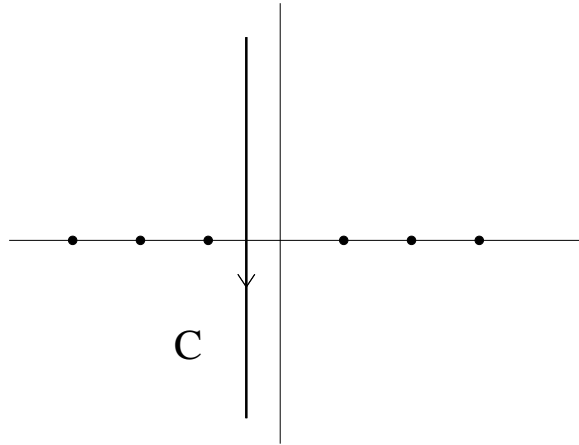


Figure 2. Integration contour in equation (2.16)

Closing the contour to the right, we recover our original power series (2.11) and (2.13) which converge for small enough x and y . Now closing the contour to the left we get power series in x and $w = \frac{1}{y}$ which converge for small x and w . We also need, as usual, to specify the branch cuts for our functions. Recall that the complexified Kähler form

$$B + iJ = \sum (B + iJ)_l e^l \quad (2.17)$$

is defined only modulo elements of integral cohomology and therefore B_l are only defined modulo an integer. In our explicit examples we will cut so that the $0 \leq B_l \leq 1$, which implies $0 \leq \arg(x) \leq 2\pi$ and $0 \leq \arg(y) \leq 2\pi$. This gives

$$\Phi_2(x, w) = - \sum_{m \geq 0} (-x)^m \sum_{n \geq 4m+1} w^n \frac{\Gamma(n)}{\Gamma(1+n+m)\Gamma(n-4m)}. \quad (2.18)$$

Clearly, $\Phi_2(x, 0) \equiv 0$ for small enough x and thus we see that zero quantum areas persist for manifolds of finite size. Note that in this example we are able to compute the volume at the conifold points exactly due to the fortuitous fact that the branch of the discriminant locus \mathcal{B} for which the three-cycles of interest to us vanish is given by a very simple toric boundary equation

$$(x, y) \in \mathcal{B} \iff |x| \leq \varepsilon, \quad 1/y = 0. \quad (2.19)$$

This feature is special to examples in which the flop involves the non-compact (line bundle) generator. Therefore it is important to establish that zero areas also occur in a case of a more conventional flop. We now turn to an example of this sort.

2.2. Calabi - Yau hypersurfaces in $\mathbb{P}^4(1, 1, 2, 2, 3)$.

The general technology here is very similar to the previous example. As a toric variety, $\mathbb{P}^4(1, 1, 2, 2, 3)$ is given by the fan which is spanned by u_1, \dots, u_5 where

$$u_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad u_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad u_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \quad u_4 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \quad u_5 = \begin{pmatrix} -3 \\ -2 \\ -2 \\ -2 \\ -1 \end{pmatrix}. \quad (2.20)$$

This manifold is singular and the singularity can be resolved by adding $u_6 = -u_1 - u_2 - u_3$ to the fan. Now the total space of the canonical line bundle is given by

$$v_0 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad v_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad v_3 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad v_4 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \quad v_5 = \begin{pmatrix} 1 \\ -3 \\ -2 \\ -2 \\ -1 \end{pmatrix}, \quad v_6 = \begin{pmatrix} 1 \\ -1 \\ -1 \\ -1 \\ 0 \end{pmatrix}. \quad (2.21)$$

The kernel of the matrix χ with columns spanned by v_0, \dots, v_6 can be chosen to have the basis $\begin{pmatrix} \xi \\ \eta \end{pmatrix}$, where

$$\begin{pmatrix} \xi \\ \eta \end{pmatrix} = \begin{pmatrix} -1 & 1 & 0 & 0 & 1 & 1 & -2 \\ -3 & 0 & 1 & 1 & -1 & -1 & 3 \end{pmatrix}. \quad (2.22)$$

The secondary fan is again generated by the two-dimensional column-vectors of (2.22) and therefore looks as shown in Fig. 3.

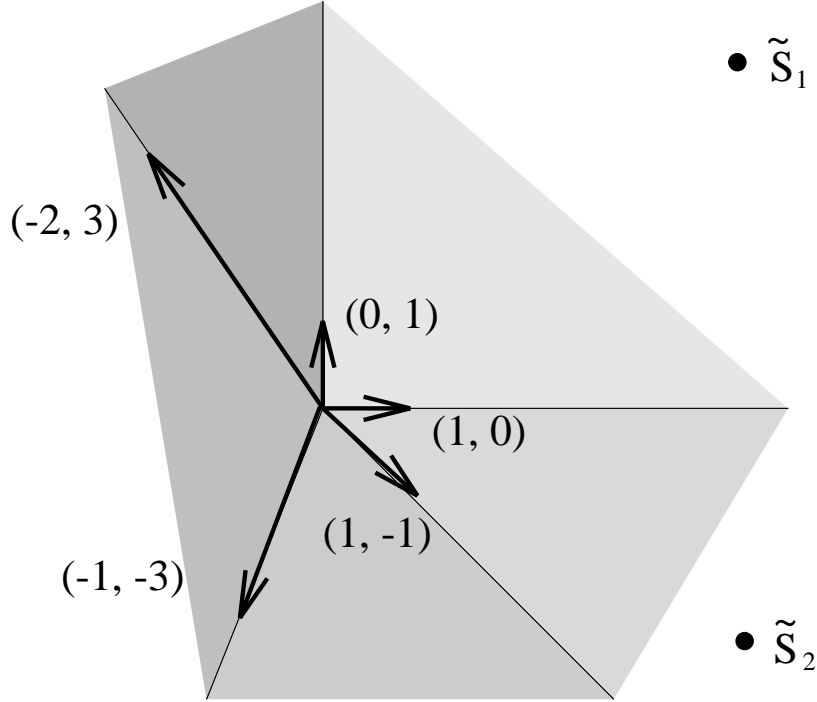


Figure 3. The secondary fan of a Calabi-Yau hypersurface in $\mathbb{P}^4(1, 1, 2, 2, 3)$

We see that again we have two smooth Calabi-Yau phases which differ by a flop. The Picard-Fuchs system is

$$\{\theta_1^2(3\theta_1 - 2\theta_2)(3\theta_1 - 2\theta_2 - 1)(3\theta_1 - 2\theta_2 - 2) - x(\theta_1 - \theta_2)^2(3\theta_1 + \theta_2 + 1) \cdots (3\theta_1 + \theta_2 + 3)\} \Phi(x, y) = 0 \quad (2.23)$$

and

$$\{\theta_2(\theta_2 - \theta_1)^2 - y(3\theta_1 + \theta_2 + 1)(3\theta_1 - 2\theta_2)(3\theta_1 - 2\theta_2 - 1)\} \Phi(x, y) = 0, \quad (2.24)$$

where coordinates x and y are chosen so that $x = y = 0$ corresponds to the infinitely large complex structure. The part of the discriminant locus we are interested in is given

in parametric form by

$$\begin{aligned} y &= \frac{(3-2s)^3}{(1-s)^2(3+s)^3} \\ x &= \frac{s(1-s)^2}{(3+s)(3-2s)^2}, \end{aligned} \quad (2.25)$$

where $|s| < \delta$ for some $\delta > 0$.

A single-valued and log-monodromy solution can be written in terms of Barnes-type integrals as

$$\Phi_0(x, y) = \sum_{n \geq 0} \frac{(-x)^n}{n!} \frac{1}{2\pi i} \int_{\mathcal{C}} \frac{\Gamma(z-n)\Gamma(3z+n+1)(-y)^z dz}{\Gamma(n-z+1)\Gamma(z+1)^2\Gamma(3z-2n+1)} \quad (2.26)$$

and

$$\Phi_2(x, y) = \sum_{n \geq 0} \frac{x^n}{n!} \frac{1}{2\pi i} \int_{\mathcal{C}} \frac{\Gamma(z-n)^2\Gamma(3z+n+1)y^z dz}{\Gamma(z+1)^2\Gamma(3z-2n+1)}, \quad (2.27)$$

where contour \mathcal{C} is shown in Fig. 4.

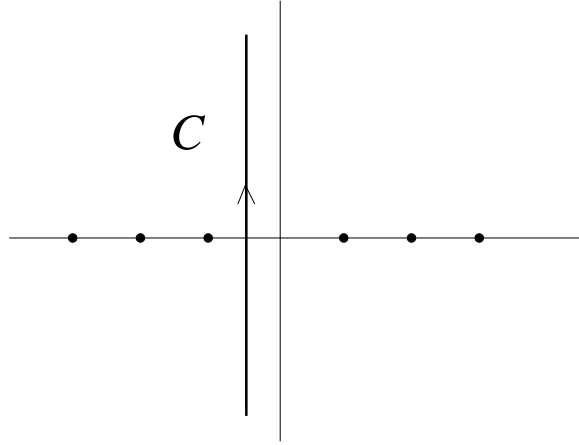


Figure 4. Integration contour in equation (2.27)

When the overall size of the manifold is held at infinity (i.e. for $x = 0$), (2.27) specializes to $\Phi_2(0, y) = \log(y)$ in agreement with results of [4] for a flop. In particular, we recover the zero area for the flopped 2-cycle for $x = 0$ and $y = 1$. However, now we can investigate whether this zero area persists when we move away from infinite radius point, i.e. for $x \neq 0$. Substituting (2.25) for the component of discriminant locus passing through $x = 0, y = 1$ into (2.27) and (2.26), we find explicit expressions for the quantum area of our two-cycle on the discriminant locus. We could not find an easy analytic proof

that this expression is identically zero. However, one can expand $\Phi_2(x(s), y(s))$ as power series in s around $s = 0$, where s is the parameter of equation (2.25). We carried out this expansion to the fifth order in s and found that the coefficients of the power series are zero. This proves (at least to that order) that the quantum volume of two-cycles remains zero for manifolds of finite overall size. As we will discuss further in section four, the fact that Φ_2 is zero on this component of the discriminant locus should follow on more general grounds from the vanishing cycle being an *integral* class, i.e. an element of $H_3(W, \mathbb{Z})$.

3. Quantum Volume

In the above discussion, we have made use of a notion of quantum volume that has been developed over the last couple of years in the context of conformal field theory and mirror symmetry. As such, it naturally gives us a notion of quantum volume for two-cycles as these are the geometrical structures which control instanton corrections. In this section, we briefly examine this notion and its extension to a definition of quantum volume for higher even-dimensional cycles.

There are many ways one can attempt to define a notion of volume in string theory. Here we shall discuss three that are interrelated and in one way or another have played a prominent role in recent physical developments.

3.1. Volume from the Linear Sigma Model

The first comes from the linear sigma model of Witten [1]. The linear sigma model is a physical realization of the symplectic quotient from classical geometry. In this way, it provides a direct link between classical and quantum geometry. To keep the discussion simple, let's consider the example of a degree eight Calabi-Yau in $W\mathbb{P}^4(1, 1, 2, 2, 2)$. This Calabi-Yau has $h^{11} = 2$ and we therefore represent the linear sigma model on this space by a $U(1)^2$ gauge theory as follows:

$$\mathcal{L} = \mathcal{L}_{\text{kin}} + \mathcal{L}_W + \mathcal{L}_{\text{gauge}} + \mathcal{L}_\theta + \sum_{a=1,2} r_a \int d^2y D_a \quad (3.1)$$

We will discuss those terms in (3.1) that are directly relevant for our purposes here. A more detailed discussion of this Lagrangian can be found in [1].

Vacuum solutions are determined from the condition that the bosonic potential energy U is zero. In our case

$$U = \frac{1}{2e_1^2} D_1^2 + \frac{1}{2e_2^2} D_2^2, \quad (3.2)$$

modulo terms whose exact form is unimportant for our discussion, where

$$D_a = -e_a^2 \left(\sum_{i=0}^6 Q_i^a |\phi_i|^2 - r_a \right), \quad a = 1, 2. \quad (3.3)$$

As discussed in [20], the charges Q_i^a can be determined from the gauge invariance of the superpotential. A convenient choice is

$$\begin{pmatrix} Q_i^1 \\ Q_i^2 \end{pmatrix} = \begin{pmatrix} -8 & 1 & 1 & 2 & 2 & 2 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & -2 \end{pmatrix}. \quad (3.4)$$

The two Fayet-Illiopoulos D-term parameters r_1 and r_2 have a direct classical geometrical interpretation. Abstractly, they specify the moment map whereby the Kähler structure on the ambient \mathbb{C}^6 induces a Kähler structure on the line bundle $\mathcal{O}(8)$, as discussed in [1]. More concretely, we see that when r_1 and r_2 are both positive, vanishing of the D-terms requires

$$|\phi_1|^2 + |\phi_2|^2 + 2|\phi_3|^2 + 2|\phi_4|^2 + 2|\phi_5|^2 - 8|p|^2 - r_1 \quad (3.5)$$

$$|\phi_1|^2 + |\phi_2|^2 - 2|\phi_6|^2 - r_2. \quad (3.6)$$

In the smooth Calabi-Yau phase, p is forced to zero by transversality, in the usual way, and we directly see that r_1 controls the size of the ambient projective space. The relevance of r_2 becomes clear when we recall that the original weighted projective space is singular at $\phi_1 = \phi_2 = 0$. For $r_2 > 0$, the second D -term ensures that ϕ_1 and ϕ_2 can not simultaneously vanish. More precisely, when $\phi_6 \neq 0$, the second $U(1)$ together with its D -term (which yield a \mathbb{C}^* action) can be used to set $\phi_6 = 1$ and recover the original Calabi-Yau, except for the singular locus. When $\phi_6 = 0$, we recover the singular locus (a quartic in $\phi_{3,4,5}$) except that the second D -term replaces the point $\phi_1 = \phi_2 = 0$ (at fixed $\phi_{3,4,5}$) with a \mathbb{CP}^1 whose radius is determined by r_2 . Thus, this classical analysis clearly shows that r_1 controls the size of the ambient (singular) weighted projective space $W\mathbb{P}^4(1, 1, 2, 2, 3)$ while r_2 controls the size of the resolving space.

In the quantum theory (in the two-dimensional field theory sense), the linear sigma model parameters will flow via the renormalization group to parameters \tilde{r}_1 and \tilde{r}_2 describing the conformally invariant nonlinear sigma model. In the language of [4] the algebraic

measure will flow to the sigma model measure, a feature that has also played an important role in the phase structure of M and F -theories on Calabi-Yau manifolds [6]. One might therefore say that the original parameters r_1 and r_2 directly measure classical sizes while \tilde{r}_1 and \tilde{r}_2 measure their string theoretic counterparts. We are working at string tree level, and hence our discussion is classical from the string perspective. If we are in type IIA string theory, though, the Kähler parameters are not renormalized by quantum string effects and we therefore can take \tilde{r}_1 and \tilde{r}_2 to be the parameters directly measuring the quantum volumes involved.

There are a few comments we should make. First, the linear sigma model is only applicable for a limited class of Calabi-Yau compactifications — those realizable in a toric setup. Thus, this definition of *classical* volume in terms of the original linear sigma model coordinates is somewhat limited and not fully intrinsic to the Calabi-Yau — it is dependent, rather, on whether and how the Calabi-Yau is being realized in an embedding space. We will return to this point in a moment. Second, we have not been careful above to distinguish between two, four and six cycles on a Calabi-Yau. The parameters r_2 and \tilde{r}_2 can be thought of as directly measuring, from the linear and nonlinear sigma model perspectives, the volume of the two-cycle involved in resolving the singularity of the Calabi-Yau space. But how do we get the volume of the whole Calabi-Yau from r_1, r_2 or \tilde{r}_1, \tilde{r}_2 ? Do we use the cubic form that arises from classical geometry? If not, what is the appropriate quantum analog? In fact, these two comments are directly related, as we will momentarily discuss.

3.2. Volume from The Non-Linear Sigma Model

A nonlinear sigma model on a Calabi-Yau M , is specified by a choice of complexified Kähler form $K = B + iJ$ and complex structure for M . If W is again the mirror to M , then in the neighborhood of a large complex structure point of W , the mirror map determines K on M to be of the form $K = t_j e^j$ [12] with

$$t_j = \frac{1}{2\pi i} \log(z_j) + \mathcal{O}(z_1, \dots, z_{h^{21}(W)}). \quad (3.7)$$

In this equation, the local coordinates on the complex structure moduli space of W are chosen so that the large complex structure point corresponds to all $z_i = 0$. The first term on the right-hand-side of (3.7) arises from the monomial-divisor mirror map of [21] and can be thought of as the leading order term to the mirror map in the limit that $\alpha' \rightarrow 0$. The other terms in (3.7) arise from instanton corrections. When there is a linear sigma model

representation of M and W , $\log(z_j)$ corresponds precisely to a Fayet-Illioupolos parameter $r_j + i\theta_j$. This matches well with our discussion above in which these complexified Fayet-Illioupolos parameters are associated with classical geometry. The instanton corrections renormalize these classical parameters to their quantum values, t_j . In particular, world sheet instantons which wrap around S^2 's directly probe these values as $t_j = \int_{S_j^2} K$ is precisely action of such a configuration.

It is worth emphasizing three points. First, if one supplies the data of a complexified Kähler class (and a complex structure) to define a nonlinear sigma model, the Kähler class directly measures quantum volumes. To reduce to a classical measure of volume one would need to judiciously take the limit as $\alpha' \rightarrow 0$. The linear sigma model and/or mirror symmetry provide systematic means for doing so. Second, the allowed choices for K — the quantum Kähler moduli space — are generally different from the classical moduli space. Some classical regions are simply unattainable while other classically forbidden regions are required. Simple examples of the former are regions in one-parameter Calabi-Yau examples with $\text{Im}(K)$ sufficiently small (e.g. on the quintic $\text{Im}(K) \geq J_0$ with $J_0 = \frac{4}{5} \sin^3(2\pi/5)$ [12]) and examples of the latter are flopped phases. Third, our discussion is at string tree level, but should be thought of as being in the context of the type IIA string on M or the type IIB on W . In either case, the moduli being discussed lie in vector multiplets and hence do not receive string loop corrections. The word quantum — initially introduced to describe two dimensional conformal field theory corrections — can then be interpreted in the full sense of quantum string theory.

So far then, we have a classical and quantum notion of two-cycle volumes; the quantum notion being the quantity of true physical relevance. How can we extend this to higher dimensional cycles? For two-cycles, the fact that string instantons probe their size gave us a direct physical probe. For higher dimensional cycles, one naturally seeks a higher dimensional probe and D-branes fit the bill.

3.3. Volume from Wrapped D-branes

In type IIB string theory on W , BPS saturated three-branes states wrapped around supersymmetric three-cycles [22] have mass given by [23]

$$M = g_5 e^{\mathcal{K}/2} |m^I F_I - n_I Z^I| \quad (3.8)$$

where g_5 is a positive constant, $\mathcal{K} = -\log(iF_I\overline{Z}^I - iZ^I\overline{F}_I)$,

$$F_I = \int_{A_I} \Omega \quad (3.9)$$

$$Z^I = \int_{B^J} \Omega \quad (3.10)$$

for some symplectic basis $\{A_I, B^J\}$ of $H_3(W, \mathbb{Z})$ and

$$n_I = \frac{1}{g_5} \int_{A_I \times S^2} \mathbf{F} \quad (3.11)$$

$$m^J = \frac{1}{g_5} \int_{B^J \times S^2} \mathbf{F}, \quad (3.12)$$

where \mathbf{F} is the RR-charge carrier five form. In the special case of, say, a magnetically neutral and singly electrically charged state, this formula shows that the three-brane mass is proportional to the period of the cycle it wraps. When a cycle collapses, therefore, we expect a new massless state to appear in the theory [24]. In this sense, these three-branes are a direct probe of the geometrical properties of three-cycles on W . As the BPS mass formula is exact and since the full quantum geometry in this sector of the theory is captured by lowest order classical calculations, wrapped three-branes of this sort probe the full quantum geometrical structure.

Of particular importance is the fact proven in [22] that for an arbitrary three cycle γ

$$\text{Vol}(\gamma) \geq \left| \int_{\gamma} \Omega \right| \quad (3.13)$$

with equality being achieved for supersymmetric three-cycles, the ones giving rise to BPS saturated states. Thus, the mass of wrapped D -3-branes directly tracks three-cycle volumes on W . Now that we have a physical observable which directly probes three-cycle volumes on W , we can use mirror symmetry, to transport this quantum geometric understanding to M . In essence, since observables like particle masses are preserved by mirror symmetry, we define the quantum volumes on M in terms of wrapped D -brane masses on M — which we directly compute from wrapped D -brane masses on W . Quantum volumes on W are thereby taken to quantum volumes on M .

To do so we first recall that the homology $H_3(W, \mathbb{Z})$ is mirror to the sum of even cycles $H_0(M, \mathbb{Z}) \oplus H_2(M, \mathbb{Z}) \oplus H_4(M, \mathbb{Z}) \oplus H_6(M, \mathbb{Z})$ on M [25] [26]. In particular, the three-cycles γ_i , $i = 1, \dots, h_W^{21}$ in $H_3(W, \mathbb{Z})$ with logarithmic periods at infinite complex

structure — points of maximal unipotent monodromy — are mirror to $H_2(M, \mathbb{Z})$. Thus, using wrapped two-branes on M or wrapped three-branes (wrapping three-cycles with the stated monodromy) on W we have states of mass

$$M = \frac{|\int_{\gamma_i} \Omega|}{(\int_W \Omega \wedge \bar{\Omega})^{\frac{1}{2}}}. \quad (3.14)$$

Up to a normalization factor $N = \frac{(\int_W \Omega \wedge \bar{\Omega})^{\frac{1}{2}}}{\int_{\gamma_0} \Omega}$, this is the absolute value of the same formula obtained earlier for the quantum volume of two-cycles on M , probed with string instantons. Thus, using N as our conversion factor from string instanton to two-brane probes, fully corrected two-branes wrapping two-cycles (as gotten from mirror symmetry) and string instantons measure the same quantum two-cycle volumes. In particular, so long as N is well behaved, strings and two-branes agree on when a two-cycle collapses. Thus, for instance, in the explicit calculations done in section 2, the center of a flop which has zero quantum volume (using string instanton probes) also has zero quantum volume using wrapped two-branes, and thus corresponds to a point with new massless states. This is one kind of mirror partner to Strominger's discussion of collapsing three-cycles in [24].

This discussion naturally leads us to consider the other three-cycles in $H_3(W, \mathbb{Z})$ which do not have logarithmic monodromy periods at infinity. These three-cycles are mirror to the other even dimensional cycles on M and following the same prescription as above, give us a definition for their quantum volume as well. Namely, consider an even dimensional integral cycle C_{even} on M , with mirror the integral three-cycle $\gamma_{C_{\text{even}}}$ on W . Then, we take the quantum volume of C_{even} to be $N|\frac{\int_{\gamma_{C_{\text{even}}}} \Omega}{\int_{\gamma_0} \Omega}|$. At large complex structure, this reduces to the classical volume as measured with the Kähler form K , but differs from it at other points in the moduli space. By construction, this definition also has the virtue of giving zero quantum volume at those points in the moduli space where we expect new massless states to arise. As in the discussion of two-cycles here as well as in [4][5], this definition of quantum volume is not monodromy invariant — this, for instance, accounts for the cuts required in section 2. Rather, it is the whole tower of BPS states which is invariant while any individual state transforms nontrivially amongst the others. Although different from what we are familiar with in classical geometry, this path dependence in the moduli space is a basic feature of quantum geometry. We briefly indicate some of the implications of this in the next section.

4. General Picture

The discussion of the previous sections leads to the following general picture. Let \mathcal{M} be the moduli space of complex structures on W (for, say, fixed and large Kähler class). We show this in figure 5.

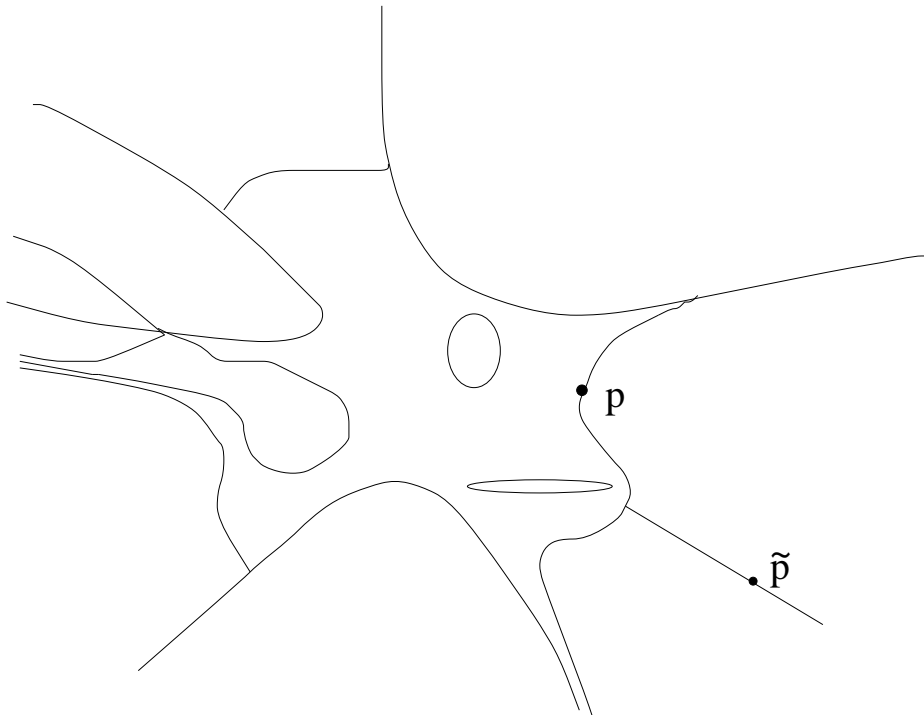


Figure 5. A typical discriminant locus within a complex structure moduli space

The curves meandering through \mathcal{M} denote the discriminant locus of W . We see that it typically has numerous components which can cross along higher codimensional loci in the moduli space. At a generic point p on any given component of the discriminant locus, some collection of three-cycles have collapsed to zero period. The mirror of this statement is that some collection of *even*-dimensional cycles on M have collapsed to zero quantum volume. Notice that in general these are not just two-cycles. The precise identification of which even-dimensional cycles have collapsed can be determined from a detailed application of mirror symmetry [27] [28]. Namely, in the neighborhood of maximal unipotent monodromy, the even cycles in $H_{2j}(M, \mathbb{Z})$ are mirror to the quotient space W_{j+1}/W_j where the W_j form the monodromy weight filtration of $H_3(M, \mathbb{Z})$. Roughly, the three-cycles in W_{j+1}/W_j have periods with \log^j type monodromy about the maximal unipotent point. More precisely,

this statement is true up to lower order *log* monodromy transformations whose precise form requires more detailed study of the mirror map than we shall undertake here.

As we move along the component of the discriminant locus on which p lies, generically the homology class of the collapsed three-cycles — being integral classes — will remain constant ¹. By construction, then, the homology class of the collapsed even-cycles on M will stay constant as well. The value of K along this component, though, will generally *change*. Recall that K depends upon the cycles with log-monodromy periods and hence only if such a cycle is amongst those which have collapsed on this component of the discriminant locus will the value of K , projected onto that cycle, remain fixed at zero.

As an example, let's consider the case of the flop studied in section 2. Using the picture just presented, the flopping two-cycle $S^2_{(j)}$ on M is mirror to a log-monodromy period three-cycle γ_j on W . Along a component of the discriminant locus of W on which this three-cycle has zero period (such as the component containing point \tilde{p} in Fig. 5), the value of t_j will be identically zero. Appropriately wrapped three-branes on W and wrapped two-branes on M will be massless. This would give a general explanation of what we found earlier: namely, flopped curves have zero quantum volume (as measured by string instantons) at the flop-point, regardless of the values of other Kähler moduli.

Along other components of the discriminant locus of W , this picture leads to the somewhat strange conclusion mentioned in the introduction regarding the collapse of a cycle vs. the collapse of its subcycles. Namely, if there are for instance components of the discriminant locus of W along which, say, the only vanishing three-cycles have periods with $\log^j, j > 1$ type monodromy (up to lower order terms, with respect to a maximal unipotent monodromy point), then in the mirror description we will have higher dimensional cycles collapsing to zero quantum volume while their lower dimensional subcycles maintain nonzero quantum volume. As we move along this component of the discriminant locus, the periods and identities of the collapsed integral three-cycles, will be unchanged. However, in general all other periods will be nonconstant and hence the quantum volumes of all other cycles on the mirror M will typically change. This includes the quantum volumes of the subcycles of the collapsed cycles. More generally, if A is a subcycle of B on M , then B can collapse to zero volume while A does not so long as the mirror period over the mirror three-cycle of B on W vanishes, but that for A does not. In any specific case, this is a question that can be answered by detailed study of the discriminant locus of W .

¹ We thank M. Gross for discussions on this point.

The simplest explicit example of this phenomenon is given by the quintic hypersurface in \mathbb{CP}^4 and W , its mirror. The discriminant locus in this case consists of a single point. As explained by Candelas, et. al [12] and utilized by Strominger [24], at the point in W

$$z_1^5 + z_2^5 + z_3^5 + z_4^5 + z_5^5 - 5\psi z_1 z_2 z_3 z_4 z_5 = 0 \quad (4.1)$$

with $\psi = 1$, one of the four three-cycles in $H_3(W, \mathbb{Z})$ collapses. The expectation initially indicated in [24] is that the mirror to this three-cycle in M is a collapsing two-cycle. This, however, is not true ². M has a single Kähler modulus with component $t_1 = B_1 + iJ_1$ with respect to the integral generator of $H_2(M, \mathbb{Z})$. At the mirror to the $\psi = 1$ point, the value of t_1 is nonzero: $t_1 \approx 1.2056$ (the normalization factor N is also finite at this point). Thus, the two-cycle has not collapsed: D -2-branes wrapping the two cycle will not become massless. Rather, using the monodromies calculated in [12], it is not hard to see that the vanishing three-cycle at $\psi = 1$, is in the homology class of the three-cycle which at infinity has period with \log^3 -type monodromy. Thus, up to lower order terms, it is the *six*-cycle on M which collapses at the mirror of the $\psi = 1$ point. Notice that in this example we only have one Kähler modulus at our disposal and by tuning it appropriately the quantum volume of the whole Calabi-Yau vanishes even though the quantum volume of the homology two-cycle does not. This accounts for us getting the desired result of a single new massless particle, even though the entire space is collapsing.

It would be of interest to further explore the properties of quantum volume we have defined. Situations in which four-cycles collapse in various ways [10][9] are prime examples in which the behavior of sub-two-cycles is important to the resulting physics. Preliminary study has shown that at least in some examples, Calabi-Yau transitions involving collapsing del Pezzo surfaces involve cycles with nonzero quantum volumes. This should allow for the verification of the conjecture made at the end of [9]. We will report on this elsewhere.

Acknowledgments

We are happy to acknowledge a number of useful discussions with Paul Aspinwall, Mark Gross, Calin Lazaroiu, David Morrison and Cumrun Vafa. BRG is supported in part by the National Science Foundation, a National Young Investigator Award and the Alfred P. Sloan Foundation. YK is supported in part by the National Science Foundation.

² After completing this work, it was brought to our attention that A. Strominger and J. Polchinski [29] have previously made the same observation.

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